THE KING'S SCHOOL

2013
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 70

Section I

10 marks

- Attempt Questions 1-10
- Answer on the Multiple Choice Answer Sheet provided.
- Allow about 15 minutes for this Section.

Section II

60 marks

- Attempt Questions 11-14
- Answer in the examination booklets provided, unless otherwise instructed.
- Start a new booklet for each question.
- Allow about 1 hour 45 minutes for this Section.

Student Number

Student Number

Section I
Questions 1 – 10 (1mark for each question)

Read each question and choose an answer A, B, C or D.
Record your answer on the Answer Sheet provided.
Allow about 15 minutes for this section.

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1 $\sin 2\alpha \cos 2\alpha =$

A) $\frac{1}{2}\sin 4\alpha$

B) $4\sin\alpha \cos\alpha$

C) $\frac{1}{2}\sin 2\alpha$

D) $2\sin 2\alpha$

2 The point R divides the interval joining P (– 3, 6) and Q (6, – 6) externally in the ratio 2:1.

Which of these are coordinates of R?

A) (3, – 2)

B) (15, – 18)

C) (0, 2)

D) (– 12, 18)

3 The polynomial $2x^3 + x - 4 = 0$ has roots α, β, and γ.

What is the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$?

A) $-\frac{1}{4}$

B) $\frac{1}{2}$

C) $-\frac{1}{2}$

D) $\frac{1}{4}$

4 Which of the following are solutions of $\frac{x}{x+1} \geq 0$?

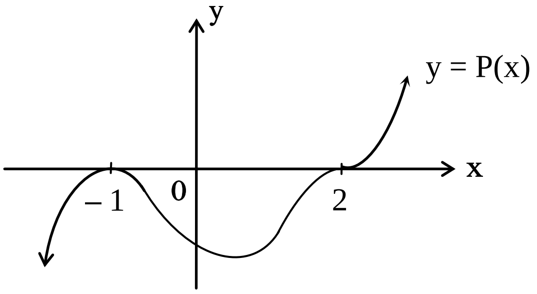
A) $x \geq 0$

B) $-1 \leq x \leq 0$

C) $x < -1$

D) $x < -1,\ x \geq 0$

5 Which of the following could be the equation of the polynomial P(x)?



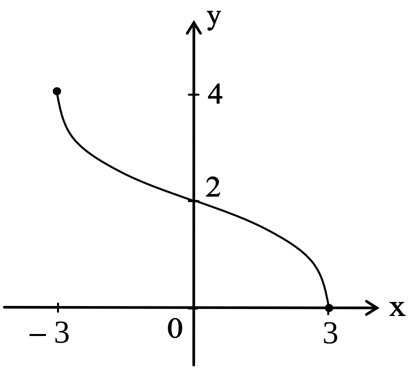
A) $P(x) = (x-1)(x+2)$

B) $P(x) = (x+1)^2(x-2)^3$

C) $P(x) = (x-1)^2(x+2)^3$

D) $P(x) = (x+1)(x-2)^3$

6 What is the equation of the function shown in the graph below?



A) $y = \dfrac{\pi}{4}\cos^{-1} 3x$

B) $y = \dfrac{4}{\pi}\cos^{-1} \dfrac{x}{3}$

C) $y = \dfrac{\pi}{4}\cos^{-1} \dfrac{x}{3}$

D) $y = \dfrac{4}{\pi}\cos^{-1} 3x$

7 A particle is moving in a simple harmonic motion with displacement x.
Its acceleration $\ddot{x}$ is given by $\ddot{x} = -4x + 3$.

What are the centre and the period of the motion?

A) $x = 3,\quad T = \dfrac{\pi}{2}$

B) $x = -3,\quad T = \pi$

C) $x = \dfrac{3}{4},\quad T = \pi$

D) $x = \dfrac{3}{4},\quad T = \dfrac{\pi}{2}$

8 What is the derivative of $y = \tan^{-1}\dfrac{1}{x^2}$?

A) $\dfrac{2}{x + x^4}$

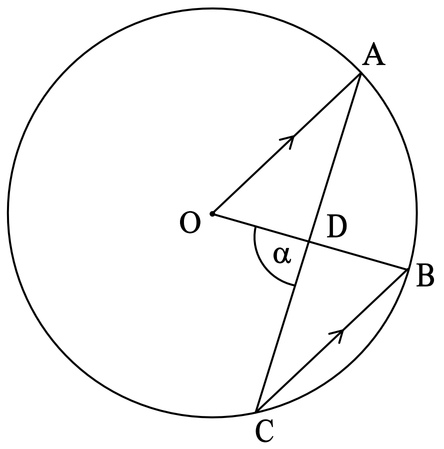
B) $\dfrac{2x}{1 + x^4}$

C) $-\dfrac{2x}{1 + x^4}$

D) $-\dfrac{2}{x + x^3}$

9 The points A, B and C lie on the circle with centre O. OA is parallel to CB. AC intersects OB at D and $\angle ODC = \alpha$.

What is the size of $\angle OAD$ in terms of α?



A) $\frac{\alpha}{2}$

B) $\frac{\alpha}{3}$

C) $\frac{2\alpha}{3}$

D) 3α

10 The trigonometric expression $\sqrt{3}\cos\theta + \sin\theta$ may be written in the form $R\cos(\theta - \alpha)$. The values of R and α could be?

(A) $R = 4$ and $\alpha = \frac{\pi}{3}$

(B) $R = 4$ and $\alpha = \frac{\pi}{6}$

(C) $R = 2$ and $\alpha = \frac{\pi}{3}$

(D) $R = 2$ and $\alpha = \frac{\pi}{6}$

End of Section I

Section II

Question 11 – 14 (15 marks each)

Allow about 1 hour 45 minutes for this section

Question 11 **MARKS**

a) Differentiate $e^{2x} \cos x$ with respect to x. 2

b) Find $\displaystyle\int \frac{dx}{\sqrt{1 - (2x)^2}}$ 3

c) Solve $\dfrac{x^2}{x^2 - 1} < 1$. 3

d) Use the substitution $u = 1 + \ln x$ to evaluate 3

$$\int_1^e \frac{dx}{x(1 + \ln x)^3} .$$

e) i) Find the coefficient of x^5 in the binomial expansion of 2

$$\left(\frac{1}{x^3} + 2x^2\right)^{15}$$

ii) What is the greatest value of n for which $\left(\dfrac{1}{x^3} + 2x^n\right)^{15}$ 1

has a non-zero term in x^5.

f) In how many ways can 10 people be divided into two groups of 5 each? 1

Question 12 **MARKS**

a) Consider the function $f(x) = \dfrac{-x}{x^2+4}$

i) Show that this function is odd. 1

ii) Find the equation of the horizontal asymptote. 1

iii) Find the coordinates of its stationary points and determine their nature. 2

iv) Sketch the graph of this function. 2

b) Let $g(x) = x - \ln(2 - x)$

i) Find the domain of $g(x)$. 1

ii) Show that $g(x)$ is monotonic increasing. 2

iii) Find the point of intersection of $y = g(x)$ and $y = x$. 1

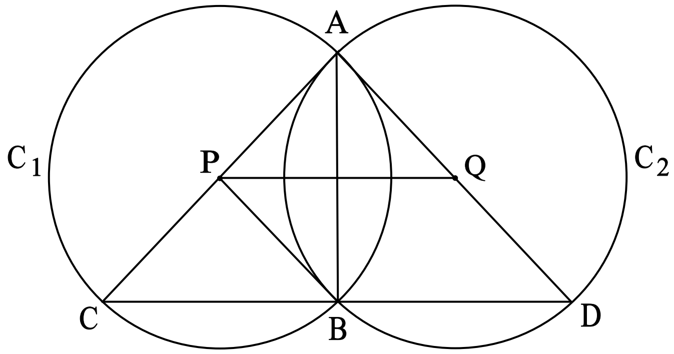
iv) On the same set of axes, sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$. 2

c) i) Given that $f(x) = (x - a)^2 g(x)$ where $f(x)$ and $g(x)$ are polynomials, show that $(x - a)$ is a factor of $f'(x)$. 1

ii) Hence, or otherwise, find the values of p and q if $x^3 - px^2 + q = 0$ has a double root at $x = 2$. 2

Question 13 **MARKS**

a) Two circles C_1 and C_2 centred at P and Q with equal radii r intersect at A and B respectively. AC is a diameter in circle C_1 and AD is a diameter in C_2.



i) Show that ΔABC is congruent to ΔABD. 1

ii) Show that $PB \parallel AD$. 2

iii) Show that PQDB is a parallelogram. 1

b) Use mathematical induction to prove that 3

$$\sum_{r=1}^{n} \ln\left(\frac{r}{r+2}\right) = \ln\left(\frac{2}{(n+1)(n+2)}\right) \text{ for } n \geq 1.$$

c) The velocity v ms^{-1} of a particle moving along the x axis is given by $v = 2x^{1.5}e^{-0.5x}$

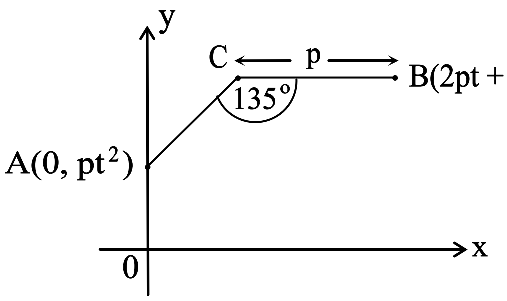
i) Find the fastest speed attained by the particle. 2

ii) After the particle reaches its maximum speed, you may consider $x = 7$ as the first approximation of the position at which its speed drops to 1 ms^{-1}. 2

By using one application of Newton's method find a better approximation of x for which the speed drops to 1 ms^{-1}.

d) $A(0, pt^2)$ and $B(2pt + p, y)$ with $p > 0$ are two points on the number plane.

The point C is chosen such that BC is parallel to x axis, $\angle ACB = 135°$ and $BC = p$ units.



i) Show that the coordinates of C are $(2pt, pt^2 + 2pt)$. 2

ii) Show that the locus of C is a parabola and find the equation of its directrix. 2

Student Number

Question 14 **MARKS**

a) Two particles are moving in a simple harmonic motion along the x axis.

The displacements of the two particles at a time t are given by
$x = a\cos 2t + 3a$ and $x = a\sin 2t$.

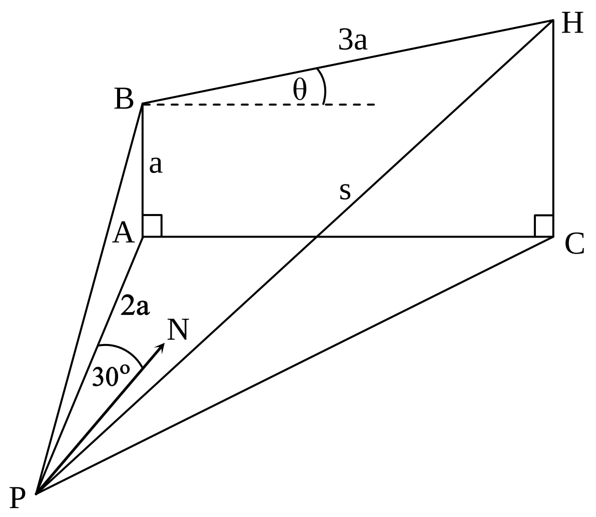
Find the closest distance between these two particles and the time at which it first occurs. 4

b) A helicopter H takes off vertically from C.

Paula and Bob are watching the helicopter. Bob is due east of the helicopter H. He is at B the top of a building AB of height a metres.

Paula is on the ground, at a point P, 2a metres away from A. The bearing of the building AB from her position at point P is 330°.

At the instant that Bob sees the helicopter at an angle of elevation of θ, its distance from him is 3a metres.



i) Show that the distance from Paula to the helicopter at this time, can be expressed as $s = a\sqrt{14 + 6\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)}$ 3

ii) It is known that at the time when θ is increasing at a rate of 0.2 radians per minute the angle of elevation θ is $\frac{\pi}{5}$ radians. 2

At what rate is s changing at this time, give your answer correct to two decimal places.

c) Joe is playing indoor soccer. He is to take a free kick from the origin.

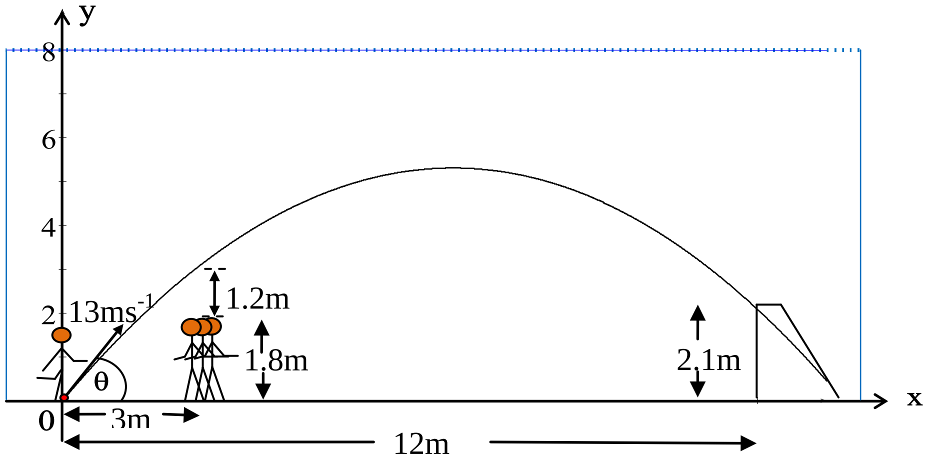
The opposing team has positioned some of its players 3 m from the ball.

These players are 1.8m tall, but can jump an extra 1.2 m to defend their goal. The ceiling of the stadium is 8m above the floor.

Joe will kick the ball with a velocity of 13 ms^{-1} at an angle of θ to the horizontal.

Use the axes as shown, and assume there is no air resistance and the position of Joe's ball t seconds after being kicked is given by the equations:

$x = 13t\cos\theta$ and $y = 13t\sin\theta - 5t^2$ (Do NOT prove this)



i) Show that the maximum height reached by the ball can be expressed as $h = \dfrac{169\sin^2\theta}{20}$. 1

ii) Show that the cartesian equation for the trajectory of the ball is $y = x\tan\theta - \dfrac{5x^2}{169}(1 + \tan^2\theta)$. 1

iii) Show that for the ball to pass over the defenders and below the ceiling the angle θ must be between $50^{\circ}41'$ and $76^{\circ}39'$. 2

iv) The goal is 2.1 m high and 12 m from the origin. 2

For what possible values of θ will the ball pass above the defensive players, below the ceiling, and enter the goal directly ?

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq 1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student Number

Multiple Choice Answer Sheet

Section I
Total marks (10)
Attempt Questions 1-10
Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

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1 (A) (B) (C) (D)

2 (A) (B) (C) (D)

3 (A) (B) (C) (D)

4 (A) (B) (C) (D)

5 (A) (B) (C) (D)

6 (A) (B) (C) (D)

7 (A) (B) (C) (D)

8 (A) (B) (C) (D)

9 (A) (B) (C) (D)

10 (A) (B) (C) (D)

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Mathematics Extension 1

Question	Algebra and Number	Differential Calculus	Functions	Geometry	Integral Calculus	Trigonometry	Total
1-10	4 / 1		2, 3, 5, 6, 8 / 5	9 /1	7 /1	1, 10 /2	/10
11	c, e, f / 6	a / 2	b / 3		d /3		/15
12			a, b, c /15				/15
13	b / 3	c(i) / 2	c(ii), d / 6	a /4			/15
14		a, b, c /15					/15
Total	/10	/19	/29	/5	/4	/2	/70

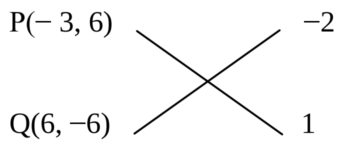
2013 Extension 1 Mathematics Solutions and Marking Guidelines

1 A

$$\sin 4\alpha = 2\sin 2\alpha \cos 2\alpha$$

$$\therefore \sin 2\alpha \cos 2\alpha = \tfrac{1}{2}\sin 4\alpha$$

2 B



$$x = \frac{1\times -3 - 2\times 6}{1-2} = 15$$

$$y = \frac{1\times 6 - 2\times -6}{1-2} = -18$$

Hence, R (15, −18)

3 D

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{1}{2} \div \frac{4}{2}$$

$$= \frac{1}{4}$$

4 D

		−1	0	
x	–	–	0	+
x + 1	–	+		+
$\frac{x}{x+1}$	+	–	0	+

From the table we can see that $\frac{x}{x+1} \geq 0$

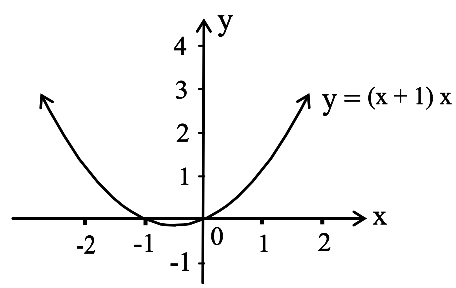
when $x < -1,\ x \geq 0$

Alternative method:

$$\frac{x}{x+1} \geq 0$$

$$(x+1)^2 \times \frac{x}{x+1} \geq 0 \times (x+1)^2 \text{ where } x \neq -1$$

$$(x+1)x \geq 0$$



From the graph, $x \geq 0,\ x \leq -1$

but as $x \neq -1$

Hence, $x < -1,\ x \geq 0$

5 B

This graph has a double root at −1, hence $(x+1)^2$ is a factor. This graph has a triple root at 2, hence $(x-2)^3$ is a factor.
The only equation which could be the correct equation is $P(x) = (x+1)^2(x-2)^3$

6 B

In A and D the domain is $-1 \le 3x \le 1$

that is $\frac{-1}{3} \le x \le \frac{1}{3}$ which does not match the given graph.

The range in B of the equation

$y = \frac{4}{\pi}\cos^{-1}\frac{x}{3}$ is $0 \le y \le 4$

Hence, B is the solution.

7 C

The acceleration $\ddot{x} = -4x + 3$

$= -4\left(x - \frac{3}{4}\right)$

The acceleration at the centre is 0.

that is $x - \frac{3}{4} = 0$ so the centre is at $x = \frac{3}{4}$

Also, $n^2 = 4$ that is $n = 2$ as $n > 0$.

Therefore, the period $= \frac{2\pi}{2} = \pi$.

Hence, C is the solution.

8 C

$y = \tan^{-1}\frac{1}{x^2}$

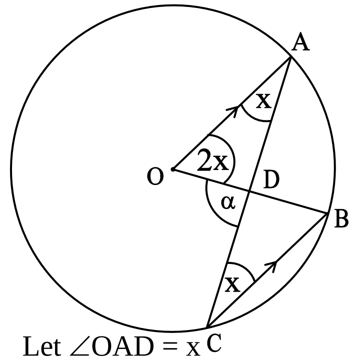
$\frac{dy}{dx} = \frac{1}{1+u^2} \times u'$ where $u = \frac{1}{x^2}$

$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{x^2}\right)^2} \times -\frac{2}{x^3}$

So $\frac{dy}{dx} = -\frac{2}{x^3\left(1+\frac{1}{x^4}\right)}$

$= -\frac{2}{x^3+\frac{1}{x}} = -\frac{2x}{1+x^4}$

9 B



Let $\angle OAD = x$

Hence $\angle ACB = x$ (alternate angles, OA is parallel to CB)

$\angle AOB = 2x$ (angle at centre is twice the angle at the circumference subtended by the same arc AB)

$\alpha = x + 2x$ (exterior angle of triangle OAD equals the sum of the two opposite interior angles)

So $\alpha = 3x$ Hence, $x = \frac{\alpha}{3}$

10 D

$R\cos\alpha = \sqrt{3}$ and

$R\sin\alpha = 1 \rightarrow R^2 = 4$

$\therefore R = 2$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = -\frac{\pi}{6}$

Question	*Suggested Marking Criteria*	*Marks*
Question 11		
a) $y = e^{2x}\cos x$ Using product rule where $u = e^{2x}$ and $v = \cos x$ $u' = 2e^{2x}$ and $v' = -\sin x$ $\frac{dy}{dx} = 2e^{2x}\cos x - e^{2x}\sin x$ $= e^{2x}(2\cos x - \sin x)$	Indicate use of Product Rule.	**1**
	State the correct solution $\frac{dy}{dx} = e2x(2\cos x - \sin x)$	**1**
b) $\int \frac{dx}{\sqrt{1-(2x)^2}} = \int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{\sqrt{4\left(\frac{1}{4}-x^2\right)}}$ $= \frac{1}{2}\int \frac{dx}{\sqrt{\frac{1}{4}-x^2}} = \frac{1}{2}\sin^{-1}2x + c$	Indicate use of sine inverse.	**1**
	Choose a correct substitution or equivalent	**1**
	State the correct solution. $\frac{1}{2}\sin^{-1}2x + C$	**1**

<table>
<tr><th>Question</th><th>Suggested Marking Criteria</th><th>Marks</th></tr>
<tr><td rowspan="3">

c) $\frac{x^2}{x^2-1} < 1$ so $\frac{x^2}{x^2-1} - 1 < 0$ that is $\frac{1}{x^2-1} < 0$

<table>
<tr><td>x</td><td></td><td>-1</td><td>1</td></tr>
<tr><td>$x^2 - 1$</td><td>+</td><td>–</td><td>+</td></tr>
</table>

From the table: $-1 < x < 1$

Alternative Method:

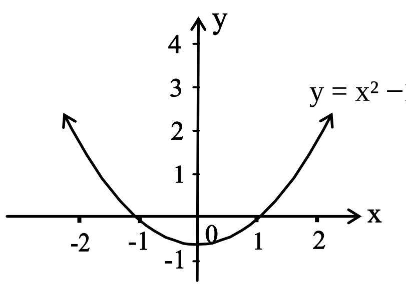
$(x^2-1)^2 \times \frac{x^2}{x^2-1} < 1 \times (x^2-1)^2$ where $x \neq \pm 1$

$(x^2-1)x^2 < x^4 - 2x^2 + 1$

$x^4 - x^2 < x^4 - 2x^2 + 1$

$x^2 - 1 < 0$

From the graph, $-1 < x < 1$



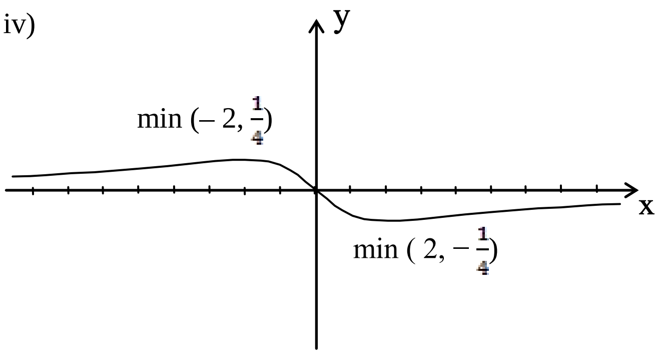
</td><td>Begin a correct method.</td><td>1</td></tr>
<tr><td>Create a correct expression without denominators.</td><td>1</td></tr>
<tr><td>State the correct solution.
$-1 < x < 1$</td><td>1</td></tr>
</table>

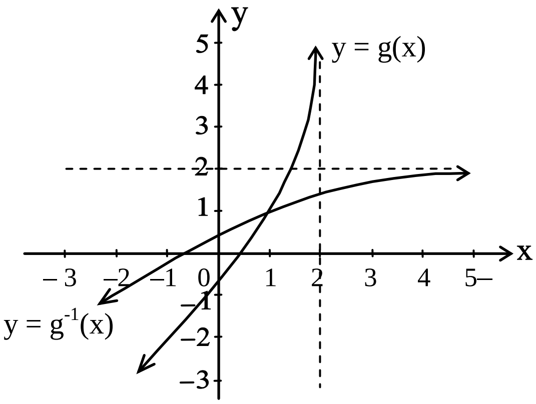
Question	*Suggested Marking Criteria*	*Marks*
d) $\int_1^e \frac{dx}{x(1+\ln x)^3}$ Let $u = 1+\ln x$, $\frac{du}{dx} = \frac{1}{x}$ $= \int_1^2 \frac{du}{u^3}$ $du = \frac{dx}{x}$ When $x = e$, $u = 2$ When $x = 1$, $u = 1$ $= \int_1^2 u^{-3} du$ $= \left[-\frac{1}{2}u^{-2}\right]_1^2 = \left[-\frac{1}{2u^2}\right]_1^2 = (-\frac{1}{8}) - (-\frac{1}{2}) = \frac{3}{8}$	Show $du = \frac{dx}{x}$ or equivalent.	**1**
	Find the new limits of integration.	**1**
	Show the correct solution. $\frac{3}{8}$	**1**
e) i) $\left(\frac{1}{x^3} + 2x^2\right)^{15}$ has $T_{k+1} = {}^{15}C_k\,(x^{-3})^{15-k}\,(2x^2)^k$ $= {}^{15}C_k\, x^{-45+3k}\, 2^k\, x^{2k} = {}^{15}C_k\, 2^k\, x^{-45+5k}$ To find the coefficient of x^5, we let $-45 + 5k = 5$ we get $k = 10$ So the term needed is ${}^{15}C_{10}\, 2^{10}\, x^5$ Hence, the coefficient of x^5 is ${}^{15}C_{10}\, 2^{10} = 3\ 075\ 072$	Show an expression for T_{k+1} (or equivalent).	**1**
	Show correct solution. ${}^{15}C_{10}\, 2^{10} = 3\ 075\ 072$	**1**
ii) $\left(\frac{1}{x^3} + 2x^n\right)^{15}$ has $T_{k+1} = {}^{15}C_k\,(x^{-3})^{15-k}\,(2x^n)^k$ $= {}^{15}C_k\, x^{-45+3k}\, 2^k\, x^{nk} = {}^{15}C_k\, 2^k\, x^{-45+(3+n)k}$ To find the value of *n* such that the term in x^5 is not a zero, we let: $-45 + (3+n)k = 5$, so $(3+n)k = 50$ $n + 3 = \frac{50}{k}$ that is $n = \frac{50}{k} - 3$ Hence, the maximum value of n occurs when k = 1and this maximum value is $n = 47$	State $n = 47$	**1**
f) The first group of 5 can be selected in ${}^{10}C_5$ ways. The other group of 5 can be formed in ${}^5C_5$ ways. So the number of ways to get the two groups of 5 is ${}^{10}C_5 \times {}^5C_5$ but the two groups formed are equal in size so they cannot be distinguished so we must divide by 2! Hence, the number of ways to form the two	State the correct answer, 126 ways.	**1**

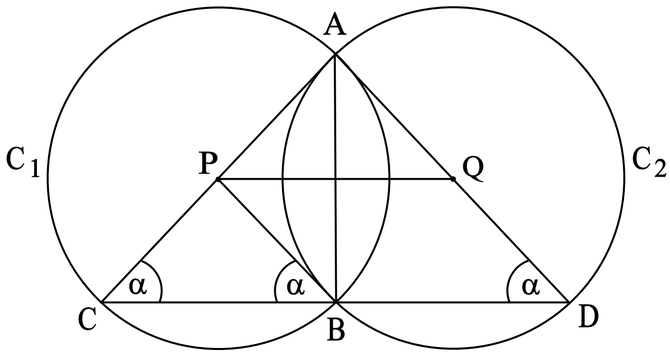
equal groups is ${}^{10}C_5 \times {}^{5}C_5 \div 2! = 126$		

<table>
<tr><th>Question</th><th>Suggested Marking Criteria</th><th>Marks</th></tr>
<tr><td colspan="3">Question 12</td></tr>
<tr><td>a) i) $f(x) = \frac{-x}{x^2+4}$
$f(-x) = \frac{-(-x)}{(-x)^2+4}$
$= \frac{x}{x^2+4} = -f(x)$
Hence $f(-x) = -f(x)$, so $f(x)$ is odd.</td><td>Show that $f(x)$ is odd.</td><td>1</td></tr>
<tr><td>ii) To find the horizontal asymptote take the limit as x approaches infinity.
$\lim_{x\to\infty} \frac{-x}{x^2+4}$
$= \lim_{x\to\infty} \frac{-\frac{x}{x^2}}{\frac{x^2}{x^2}+\frac{4}{x^2}}$
$= \lim_{x\to\infty} \frac{-\frac{1}{x}}{1+\frac{4}{x^2}}$
$= \frac{0}{1+0} = 0$
So the horizontal asymptote is $y = 0$</td><td>Show the correct solution. Horizontal asymptote is $y = 0$</td><td>1</td></tr>
<tr><td rowspan="2">iii) $f(x) = \frac{-x}{x^2+4}$ let $u = -x$ and $v = x^2+4$
$u' = -1$ and $v' = 2x$
$f'(x) = \frac{-(x^2+4)-(-x)2x}{(x^2+4)^2} = \frac{-x^2-4+2x^2}{(x^2+4)^2}$
$= \frac{x^2-4}{(x^2+4)^2}$
let $f'(x) = 0$ to find the possible stationary turning points ,we get $\frac{x^2-4}{(x^2+4)^2} = 0$ that is
$x = \pm 2$.
when $x = 2$, $y = -\frac{1}{4}$ and $x = -2$, $y = \frac{1}{4}$
<table>
<tr><td>x</td><td>-3</td><td>-2</td><td>0</td><td>2</td><td>3</td></tr>
<tr><td>$f'(x)$</td><td>$\frac{5}{169}$</td><td>0</td><td>$-\frac{1}{4}$</td><td>0</td><td>$\frac{5}{169}$</td></tr>
<tr><td>$f(x)$</td><td>/</td><td>—</td><td>\</td><td>—</td><td>/</td></tr>
</table>
Max T.P. at $(-2, \frac{1}{4})$ Min T.P. at $(2, -\frac{1}{4})$</td><td>Find 1 stationary point and determine its nature or find the x value for both turning points.</td><td>1</td></tr>
<tr><td>Find the second stationary point and determine its nature or find the y value for both turning points and determine their nature.

Max T.P. at $(-2, \frac{1}{4})$ Min T.P. at $(2, -\frac{1}{4})$</td><td>1</td></tr>
</table>

Question	*Suggested Marking Criteria*	*Marks*
iv) 	Sketch a curve, correctly showing their stationary points.	**1**
	Sketch a curve, correctly showing their curve approaching their asymptotes.	**1**
b) i) $g(x) = x - \ln(2 - x)$ g(x) exists if $\ln(2 - x)$ exists that is if $2 - x > 0$ so $-x > -2$ that is $x < 2$. Hence, the domain of g(x) is $x < 2$	State the correct domain, g(x) is $x < 2$.	**1**
ii) $g'(x) = 1 - \frac{-1}{2-x}$ $= 1 + \frac{1}{2-x}$ as $x < 2$, $2 - x > 0$ that is $g'(x) > 0$ as it is the sum of two positive terms. Therefore, g(x) is monotonic increasing throughout the domain of the function.	Calculate the correct derivative. $g'(x) = 1 + \frac{1}{2-x}$	**1**
	Justify that the curve is monotonic increasing. $g'(x) > 0$ for all x as $2 - x > 0$	**1**
iii) To find the point of intersection of y= x and y = g(x) solve x = g(x) solve $x = x - \ln(2 - x)$ $\ln(2 - x) = 0$ so $2 - x = 1$ $x = 1$ and so $y = 1$ Hence, the point of intersection is (1, 1).	Find the correct point of intersection. (1, 1)	**1**

Question	*Suggested Marking Criteria*	*Marks*
iv) Since g(x) is a monotonic increasing function then to sketch the graph of g(x) we must first find g''(x) to determine the concavity of this curve. $g''(x)=\frac{1}{(2-x)^2} > 0$ for all $x < 2$. Hence, the curve is strictly increasing and is concave up for all $x < 2$.	Sketch y = g(x)	**1**
	Sketch y = g-1(x)	**1**
(c) (i) $f(x) = (x-a)^2\, g(x)$ $f'(x) = 2(x-1)g(x) + g'(x)(x-a)^2$ $\therefore\ f'(x) = (x-1)[2g(x) + g'(x)(x-1)]$ $\therefore$ $(x-a)$ is a factor of $f'(x)$	Correct answer	**1**
(ii) $8 - 4p + q = 0$ $12 - 4p = 0$	Finding equations to solve simultaneously	**1**
$\therefore$ $q = 4$ $p = 3$	Correct answer	**1**

Question	*Suggested Marking Criteria*	*Marks*
Question 13		
a)  i) In ΔABC and ΔABD, AC = AD (diameters of equal circles) $\angle ABC = \angle ABD = 90^{\circ}$ (angles in semicircles with diameters AC and AD) AB is common $\therefore \Delta ABC \equiv \Delta ABD$ (RHS)	Show that ΔABC is congruent to ΔABD.	**1**
ii) Let $\angle ADB = \alpha$ $\angle ACB = \alpha$ (corresponding angles of congruent triangles ABC and ABD are equal) PB = PC (equal radii of circle C_1) $\therefore \Delta PBC$ is isosceles (2 equal sides) $\therefore \angle PBC = \alpha$ (base angles of isosceles ΔPBC are equal) $\therefore \angle ADB = \angle PBC = \alpha$ As these two angles are corresponding and equal hence PB $\parallel QD$ (converse theorem of corresponding angles)	Show $\angle PBC = \angle PCB$.	**1**
	Show that PB $\parallel$ AD.	**1**
iii) PB $\parallel QD$ (proved above) PB = QD (equal radii of equal circles) Hence PQDB is a parallelogram as it has one pair of opposite sides that are parallel and equal.	Show that PQDB is a parallelogram.	**1**

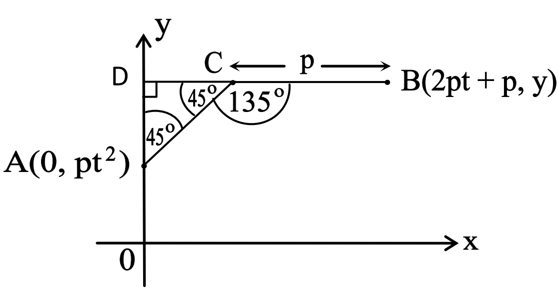
Question	*Suggested Marking Criteria*	*Marks*
b) Prove that $\sum_{r=1}^{n} \ln\left(\frac{r}{r+2}\right) = \ln\left(\frac{2}{(n+1)(n+2)}\right)$ for $n \geq 1$ For n = 1 LHS = $\ln\left(\frac{1}{3}\right)$ and RHS = $\ln\left(\frac{2}{2\times 3}\right) = \ln\left(\frac{1}{3}\right)$ Since LHS = RHS, the statement is true for n = 1 Assume the statement is true for n = k that is $\ln\left(\frac{1}{3}\right)+\ln\left(\frac{2}{4}\right)+....+\ln\left(\frac{k}{k+2}\right)$ $= \ln\left(\frac{2}{(k+1)(k+2)}\right)$	Prove statement true for n = 1.	**1**
Our aim is to prove the statement is true for n = k+1 that is $\ln\left(\frac{1}{3}\right)+\ln\left(\frac{2}{4}\right)+....+\ln\left(\frac{k}{k+2}\right)+\ln\left(\frac{k+1}{k+3}\right)$ $=\ln\left(\frac{2}{(k+2)(k+3)}\right)$ LHS = $\ln\left(\frac{1}{3}\right)+....+\ln\left(\frac{k}{k+2}\right)+\ln\left(\frac{k+1}{k+3}\right)$ $= \ln\left(\frac{2}{(k+1)(k+2)}\right)+\ln\left(\frac{k+1}{k+3}\right)$ (from assumption)	Assume true for n=k, attempt to prove true for n = k+1	**1**
$= \ln\left(\frac{2}{(k+1)(k+2)} \times \frac{k+1}{k+3}\right)$ $= \ln\left(\frac{2}{(k+2)(k+3)}\right) =$ RHS Hence, if the statement is true for n = k then it is also true for n = k + 1. Now, the statement is true for n =1 and hence by mathematical induction it is true for n = 2, 3 and so on. Hence the statement is true for all positive integers $n \geq 1$.	Satisfactory completion of proof.	**1**

<table>
<tr><th>Question</th><th>Suggested Marking Criteria</th><th>Marks</th></tr>
<tr><td rowspan="2">
c) i) $v = 2x^{1.5}e^{-0.5x}$ So $v^2 = 4x^3e^{-x}$
$\frac{1}{2}v^2 = 2x^3e^{-x}$
using the product rule
let $u = 2x^3$ and $v = e^{-x}$
$u' = 6x^2$ and $v' = -e^{-x}$
$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 6x^2e^{-x} - 2x^3e^{-x}$

$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 2x^2e^{-x}(3-x)$
To find the fastest speed we let $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 0$
So $2x^2e^{-x}(3-x) = 0$ that is when x = 0 or 3

<table>
<tr><td>x</td><td>0</td><td>1</td><td>3</td><td>4</td></tr>
<tr><td>$\frac{dv}{dx}$</td><td>0</td><td>+</td><td>0</td><td>−</td></tr>
<tr><td>v</td><td>—</td><td>/</td><td>—</td><td>\</td></tr>
</table>
From the table the maximum speed occurs when x = 3
Fastest speed $v = 2 \times 3^{1.5} \times e^{-1.5}$
$= 2.3188\ldots ms^{-1}$]
</td>
<td>Calculate the correct function for the acceleration.
$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 2x^2e^{-x}(3-x)$</td>
<td>1</td></tr>
<tr><td>Find the velocity when their derivative indicates a maximum stationary point.

$v = 2.3188\ldots ms^{-1}$</td>
<td>1</td></tr>
</table>

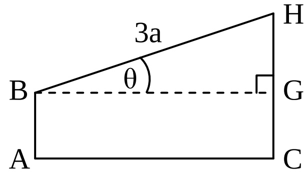
Question	*Suggested Marking Criteria*	*Marks*
ii) We need to find v = 1 that is we need to solve $2x^{1.5}e^{-0.5x} = 1$ which is $2x^{1.5}e^{-0.5x} - 1 = 0$ Let g(x) = $2x^{1.5}e^{-0.5x} - 1$ as x = 7 is the first approximation of the root then $x_1 = 7 - \frac{g(7)}{g'(7)}$ Using the product rule let $u = 2x^{1.5}$ and $v = e^{-0.5x}$	Correctly substitute into the equation for Newton's Method.	**1**
$u' = 3x^{0.5}$ and $v' = -0.5e^{-0.5x}$ So $g'(x) = 3x^{0.5}e^{-0.5x} - x^{1.5}e^{-0.5x}$ $= x^{0.5}e^{-0.5x}(3 - x)$ $\therefore x_1 = 7 - \frac{2\times 7^{1.5}\times e^{-3.5} - 1}{7^{0.5}\times e^{-3.5}(3-7)}$ = 7 + 0.37088… So x_1 = 7.37088… Note: when x = 7 , v = 1.1185... and when x = 7.37088… v = 1.00401... This shows that x = 7.37088…is a better approximation.	Calculate the correct value for the second approximation. x = 7.37088	**1**
d) i)  The x coordinate of C is x of B less by p. ∴ x of C = 2pt + p − p = 2pt	Show the x value at C is 2pt.	**1**
Now producing the horizontal line BC to meet the y axis at a right angle at D. We are given ∠ BCA = 135° then ∠ACD = 45° (angle of a straight line BD) ∴∠DAC = 45° (angle sum of ΔADC) So ΔACD is right angled and isosceles. ∴AD = DC = 2pt Hence, the y coordinate of C is the y coordinate of A plus 2pt. y of C = $pt^2 + 2pt$ Hence, the coordinate of are C (2pt , pt^2 + 2pt).	Show the y value at C is pt^2 + 2pt.	**1**

Question	*Suggested Marking Criteria*	*Marks*
ii) To find the equation of the locus of C, we need to find a relationship between x and y independent of t. Given that $x = 2pt$ and $y = pt^2 + 2pt$ $\therefore\ t = \frac{x}{2p}$ so $y = p \times \left(\frac{x}{2p}\right)^2 + 2p \times \frac{x}{2p}$ $y = p \times \frac{x^2}{4p^2} + x$ $y = \frac{x^2}{4p} + x$ $4py = x^2 + 4px$ $x^2 + 4px + 4p^2 = 4py + 4p^2$ $(x + 2p)^2 = 4p(y + p)$ This is the equation of a parabola with vertex at $(-2p, -p)$ and focal length p. Hence, its directrix has the equation $y = -2p$.	Correctly substitute the expression for t into the expression for y. $y = \frac{x^2}{4p} + x$	**1**
	Find the equation of the directrix. $y = -2p$	**1**

<table>
<tr><th>Question</th><th>Suggested Marking Criteria</th><th>Marks</th></tr>
<tr><td colspan="3">Question 14</td></tr>
<tr><td rowspan="4">
a) The distance between the particles can be expressed as D = a cos2t + 3a – a sin2t.
a cos2t – a sin2t can be expressed in the form
r cos (2t + α) where $r=\sqrt{a^2+a^2}=a\sqrt{2}$
and $\tan\alpha=\frac{a}{a}=1$ that is $\alpha=\frac{\pi}{4}$ as α is an acute angle.
Therefore D = $a\sqrt{2}\cos\left(2t+\frac{\pi}{4}\right)+3a$
As the minimum of $a\sqrt{2}\cos\left(2t+\frac{\pi}{4}\right)$
is $-a\sqrt{2}$ then the minimum distance between the particles is
$D=-a\sqrt{2}+3a=a(3-\sqrt{2})$ and this occurs for first time when
$t=-\frac{\pi}{8}+\frac{\pi}{2}=\frac{3\pi}{8}$
Alternative method:
D = a cos2t + 3a – a sin2t
$\frac{dD}{dt}=-2a\sin 2t-2a\cos 2t$
To find the minimum distance between the particles
we let $\frac{dD}{dt}=0$ that is
$-2a\sin 2t-2a\cos 2t=0$
$\sin 2t+\cos 2t=0$
$\tan 2t=-1=\tan(-\frac{\pi}{4})$
$2t=-\frac{\pi}{4},\frac{3\pi}{4},\frac{7\pi}{4}\ldots$
$t=\frac{3\pi}{8},\frac{7\pi}{8}\ldots\text{as } t>0$
Use 2nd derivative to determine max or min.
$\frac{d^2D}{dt^2}=-4a\cos 2t+4a\sin 2t$
<table>
<tr><td>t</td><td>$\frac{3\pi}{8}$</td><td>$\frac{7\pi}{8}$</td></tr>
<tr><td>$\frac{d^2D}{dt^2}$</td><td>positive</td><td>Negative</td></tr>
</table>
From the table the minimum distance occur for the first time when $t=\frac{3\pi}{8}$
$D=a\times\frac{-1}{\sqrt{2}}+3a-a\times\frac{1}{\sqrt{2}}=a(3-\sqrt{2})$
Hence, the minimum distance between the points is $(3-\sqrt{2})$ a.
</td>
<td>Find the expression for the distance between the particles.
D = a cos2t + 3a – a sin2t.</td><td>1</td></tr>
<tr><td>Differentiate the expression for the distance between the particles and find when it equals zero.</td><td>1</td></tr>
<tr><td>Determine the time when the distance between the particles is a minimum.
$t=\frac{3\pi}{8}$</td><td>1</td></tr>
<tr><td>Correctly calculate the closest distance between the particles.
$d=(3-\sqrt{2})$ a.</td><td>1</td></tr>
</table>

<table>
<tr><th>Question</th><th>Suggested Marking Criteria</th><th>Marks</th></tr>
<tr><td rowspan="3">

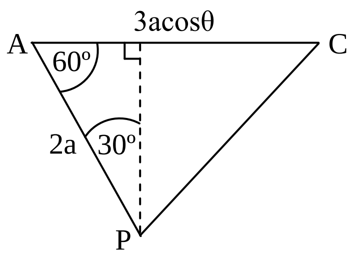
b) i)



$\cos\theta = \frac{BG}{3a}$ $\quad\quad$ $\sin\theta = \frac{HG}{3a}$

$\therefore AC = BG = 3a\cos\theta$ $\quad\quad$ $HG = 3a\sin\theta$

$\therefore HC = a + 3a\sin\theta$



Using the cosine rule in triangle APC, we get:

$PC^2 = 4a^2 + 9a^2\cos^2\theta - 12a^2\cos\theta\cos 60^\circ$

$= 4a^2 + 9a^2\cos^2\theta - 6a^2\cos\theta$

Using Pythagoras theorem in triangle PCH, we get:

$PH^2 = PC^2 + HC^2$

$s^2 = 4a^2 + 9a^2\cos^2\theta - 6a^2\cos\theta + a^2 + 6a^2\sin\theta + 9a^2\sin^2\theta$

$= 4a^2 + 9a^2(\cos^2\theta + \sin^2\theta) - 6a^2\cos\theta + a^2 + 6a^2\sin\theta$

$= 14a^2 + 6a^2\sin\theta - 6a^2\cos\theta$

$= a^2(14 + 6\sin\theta - 6\cos\theta)$

$6\sin\theta - 6\cos\theta$ can be expressed in the form

$r\sin(\theta - \alpha)$ where $r = \sqrt{6^2 + 6^2} = 6\sqrt{2}$

and $\tan\alpha = \frac{6}{6} = 1$ that is $\alpha = \frac{\pi}{4}$ as α is an acute angle.

So $s^2 = a^2(14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4}))$ $\quad\quad$ Hence $s = a\sqrt{14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})}$ as $s > 0$

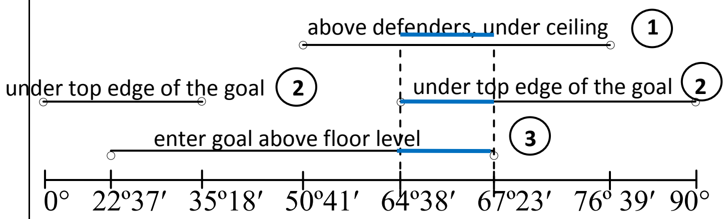
</td><td>Show that $BG = 3a\cos\theta$</td><td>1</td></tr>
<tr><td>Show that $HC = a + 3a\sin\theta$</td><td>1</td></tr>
<tr><td>Show that $s = a\sqrt{14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})}$</td><td>1</td></tr>
</table>

Question	*Suggested Marking Criteria*	*Marks*
ii) $\frac{ds}{dt} = \frac{ds}{d\theta} \times \frac{d\theta}{dt}$ $s = a\sqrt{14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})}$ $s = a\left(14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})\right)^{\frac{1}{2}}$ $\frac{ds}{d\theta} = \frac{a}{2} \times 6\sqrt{2}\cos(\theta - \frac{\pi}{4})\left(14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})\right)^{\frac{-1}{2}}$ Therefore $\frac{ds}{d\theta} = \frac{3\sqrt{2}\,a\cos(\theta - \frac{\pi}{4})}{\sqrt{14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})}}$ but $\frac{d\theta}{dt} = 0.2$ So $\frac{ds}{dt} = \frac{3\sqrt{2}\,a\cos(\theta - \frac{\pi}{4})}{\sqrt{14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})}} \times 0.2$ (rate for any θ) As $\theta = \frac{\pi}{5}$ then $\frac{ds}{dt} = \frac{3\sqrt{2}\,a\cos(\frac{\pi}{5} - \frac{\pi}{4})}{\sqrt{14 + 6\sqrt{2}\sin(\frac{\pi}{5} - \frac{\pi}{4})}} \times 0.2$ $\frac{ds}{dt} = \frac{3\sqrt{2}\,a\cos(-\frac{\pi}{20})}{\sqrt{14 + 6\sqrt{2}\sin(-\frac{\pi}{20})}} \times 0.2$ $\frac{ds}{dt} = 1.1771265\ldots \times 0.2. \times a$ $\frac{ds}{dt} \approx 0.24a$ units per minute.	Find $\frac{ds}{d\theta}$. $\frac{ds}{d\theta} = \frac{3\sqrt{2}\,a\cos(\theta - \frac{\pi}{4})}{\sqrt{14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})}}$	**1**
	Calculate the correct expression for $\frac{ds}{dt}$. $\frac{ds}{dt} = \frac{3\sqrt{2}\,a\cos(\theta - \frac{\pi}{4})}{\sqrt{14 + 6\sqrt{2}\sin(\theta - \frac{\pi}{4})}} \times 0.2$	**1**
c) i) $y = 13t\sin\theta - 5t^2$ So $\dot{y} = 13\sin\theta - 10t$ To find the time taken by the ball to reach the maximum height we let $\dot{y} = 0$, we get: $13\sin\theta - 10t = 0$ $13\sin\theta = 10t$ $t = \frac{13}{10}\sin\theta$ Now, the maximum height is $y_{max} = 13 \times \frac{13}{10}\sin\theta \times \sin\theta - 5 \times \frac{169}{100}\sin^2\theta$ $= \frac{169}{10}\sin^2\theta - \frac{169}{20}\sin^2\theta$ $= \frac{169}{20}\sin^2\theta$	Show that the maximum height can be expressed as $h = \frac{169\sin^2\theta}{20}$.	**1**

Question	*Suggested Marking Criteria*	*Marks*
ii) $x = 13t\cos\theta$ so $t = \frac{x}{13\cos\theta}$ As $y = 13t\sin\theta - 5t^2$ then $y = \frac{13\sin\theta\ x}{13\cos\theta} - 5\times\frac{x^2}{169\cos^2\theta}$ $y = x\tan\theta - \frac{5x^2}{169}\sec^2\theta$ Hence $y = x\tan\theta - \frac{5x^2}{169}(1 + \tan^2\theta)$	Show that the cartesian equation is $y = x\tan\theta - \frac{5x^2}{169}(1+\tan^2\theta)$.	**1**

Question	*Suggested Marking Criteria*	*Marks*
iii) For the ball to pass under the ceiling, if the maximum height $y_{max} < 8$ that is $\frac{169}{20}\sin^2\theta < 8$ $\sin^2\theta < \frac{160}{169}$ $\frac{-4\sqrt{10}}{13} < \sin\theta < \frac{4\sqrt{10}}{13}$ But θ is an acute angle so $\sin\theta > 0$ that is $0 < \sin\theta < \frac{4\sqrt{10}}{13}$ Now as sin θ is an increasing function for θ between 0° and 90° then $0^0 < \theta < 76^\circ 39'$ (A)	Calculate the solution for the maximum height to be less than 8m. $0^0 < \theta < 76^\circ 39'$	**1**
The ball must not be stopped by the defenders. This occur if when x = 3, y > 3 that is $3\tan\theta - \frac{45}{169}(1+\tan^2\theta) > 3$ $507\tan\theta - 45 - 45\tan^2\theta - 507 > 0$ $45\tan^2\theta - 507\tan\theta + 552 < 0$ First we solve $45\tan^2\theta - 507\tan\theta + 552 = 0$, we get: $\tan\theta = \frac{507 \pm \sqrt{(-507)^2 - 4\times 45\times 552}}{90}$ $= 1.22110\ldots, 10.04556..$ As $y = 45\tan^2\theta - 507\tan\theta + 552$ is a parabola below the tan θ axis when tanθ is between 1.22110… and 10.04556… So, the solution of $45\tan^2\theta - 507\tan\theta + 552 < 0$ is $1.22110\ldots < \tan\theta < 10.04556\ldots$ As tan θ is an increasing function for θ between 0° and 90° then $50^\circ 41' < \theta < 84^\circ 19'$ (B) Now, for the ball to pass over the defenders, but under the ceiling, it must satisfy both conditions (A) and (B) Hence $50^\circ 41' < \theta < 76^\circ 39'$ (1)	Calculate the solution for the ball to stay above the defenders. $50^\circ 41' < \theta < 76^\circ 39'$	**1**

Question	*Suggested Marking Criteria*	*Marks*
iv) For the ball to enter the goal When x = 12, y < 2.1 that is $12\tan\theta - \frac{720}{169}(1+\tan^2\theta) < 2.1$ $12\tan\theta - \frac{720}{169}(1+\tan^2\theta) < 2.1$ $2028\tan\theta - 720 - 720\tan^2\theta - 354.9 < 0$ $720\tan^2\theta - 2028\tan\theta + 1074.9 > 0$ First ,we solve: $720\tan^2\theta - 2028\tan\theta + 1074.9 = 0$ we get: $\tan\theta = \frac{2028 \pm \sqrt{(-2028)^2 - 4\times720\times1074.9}}{1440}$ $= 0.707986\ldots$ or $2.10868\ldots.$ As $y = 720\tan^2\theta - 2028\tan\theta + 1074.9$ is a parabola above the $\tan\theta$ axis when $\tan\theta < 0.707986\ldots$, $\tan\theta > 2.10868\ldots$ As $\tan\theta$ is an increasing function for θ between 0° and 90° then $\theta < 35°18'$, $\theta > 64°38'$ 2	Calculate the solution for the ball to enter the goal. $\theta < 35°18'$ or $\theta > 64°38$ and/or $22°37' < \theta < 67°23'$	**1**
Now, For the ball to enter the goal directly, it must be above ground at x = 12 that is When x = 12, y > 0 so $12\tan\theta - \frac{720}{169}(1+\tan^2\theta) > 0$ $2028\tan\theta - 720 - 720\tan^2\theta > 0$ $720\tan^2\theta - 2028\tan\theta + 720 < 0$ First, we solve: $720\tan^2\theta - 2028\tan\theta + 720 = 0$ we get: $\tan\theta = \frac{2028 \pm \sqrt{(-2028)^2 - 4\times720\times720}}{1440}$ $= \frac{5}{12}$ or $\frac{12}{5}$ As $y = 720\tan^2\theta - 2028\tan\theta + 720$ is a parabola below the $\tan\theta$ axis when $\frac{5}{12} < \tan\theta < \frac{12}{5}$ As $\tan\theta$ is an increasing function for θ between 0° and 90° then $22°37' < \theta < 67°23'$ 3 (continues on next page)	State the correct final range of angles for θ. $64°38' < \theta < 67°23'$	**1**

Question	*Suggested Marking Criteria*	*Marks*
(continued) Hence all of three conditions must hold simultaneously for a goal to be scored.  It can be seen on the above diagram that the largest range of values for which all of the conditions hold is $64^\circ38' < \theta < 67^\circ23'$.		